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Multidimensional infinitely divisible cascades

Application to the modelling of intermittency in turbulence

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Abstract. The framework of infinitely divisible scaling was first developed to analyse the statistical intermittency of turbulence in fluid dynamics. It also reveals a powerful tool to describe and model various situations including Internet traffic, financial time series, textures ... A series of recent works introduced the infinitely divisible cascades in 1 dimension, a family of multifractal processes that can be easily synthesized numerically. This work extends the definition of infinitely divisible cascades from 1 dimension to d dimensions in the scalar case. Thus, a class of models is proposed both for data analysis and for numerical simulation in dimension $d \geq 1$. In this article, we give the definitions and main properties of infinitely divisible cascades in d dimensions. Then we focus on the modelling of statistical intermittency in turbulent flows. Several other applications are considered.

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1 Introduction

Scale invariance is now considered as a well-known property of a wide variety of systems ranging from turbulent flows [22] to Internet traffic [37], DNA series [2,38] or natural images [47]. The statistical modelling of turbulent fluid flows has certainly been one of the most challenging and stimulating problems. It gave birth to a variety of tools as far as the understanding of scale invariance is concerned [22]. A usual evidence for scale invariance is the observation of a power-law power spectrum. In turbulence, we think of the famous $k^{-5/3}$ velocity spectrum. Another property of turbulent flows is their departure from Gaussian distributions. The intermittency phenomenon is often defined from a statistical viewpoint as the evolution of the probability density functions (pdf) of increments or wavelet coefficients from nearly Gaussian at larger scales to far from Gaussian at smaller scales. One would therefore appreciate to have non Gaussian scale invariant stochastic processes at hand. Hopefully, multifractal processes meet both properties. But there are not so many well known families of processes.

In one dimension, recent works [4,7,17,18,34,44] have given very interesting results by defining the class of infinitely divisible cascades including the subclass of compound Poisson cascades. The purpose of this paper is the generalization of these definitions in a d-dimensional space ($d \geq 2$). For instance, this will provide us with relevant models for scalar quantities such as the 3-dimensional dissipation field $\varepsilon(\mathbf{x})$ in turbulent flows or the 2-dimensional intensity $I(\mathbf{x})$ of grey levels images.

Note that we will consider scalar multifractal processes only, not vectorial. From a mathematical point of view, the quantity of interest can be seen as the density of a positive measure: the turbulent dissipation field is always non-negative ($\varepsilon(\mathbf{x}) \geq 0$) and the intensity of an image is also non negative ($I(\mathbf{x}) \geq 0$). In the multifractal framework, the term scale invariance then refers to the power law behaviour of the moments of some scale dependent quantity[1] built on the process X under study. For a positive scalar process $X(\mathbf{x})$ defined on $\mathbb{R}^d$, one often uses the box averages over a ball of radius r and volume V_r

$$\varepsilon_r(\mathbf{x}) = \frac{1}{V_r} \int_{\|\mathbf{x}'-\mathbf{x}\|<r} X(\mathbf{x}')\, d\mathbf{x}'. \tag{1}$$

In short, scale invariance is then described by a set of multifractal exponents $\tau(q)$ defined through:

$$\mathbb{E}\varepsilon_r(\mathbf{x})^q \propto r^{\tau(q)}, \tag{2}$$

where $\mathbb{E}$ denotes mathematical expectation. For a given process $X(\mathbf{x})$, the multifractal formalism is said to be verified when the multifractal spectrum $D(h)$ is related to

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[1] For instance increments $X(\mathbf{y})-X(\mathbf{x})$ in function of $\|\mathbf{y}-\mathbf{x}\|$ or wavelet coefficients $T_X(\mathbf{x}, a)$ at scale a.

the scaling exponents $\tau(q)$ through a Legendre transform:

$$D(h) = \inf_q [d + qh - \tau(q)]. \qquad (3)$$

The multifractal spectrum $D(h)$ quantifies the relative importance of singularities associated to local Hölder regularity h in $X(\mathbf{x})$ [23,22,5,25,26,41]. One step further, the property of Extended Self-Similarity (ESS) was introduced in the study of turbulent flows in the early 90s [22]. At first, it was used to increase the precision of scaling exponent estimate. It relates moments of different orders through a relative scaling behaviour:

$$\mathbb{E}\varepsilon_r^q \propto (\mathbb{E}\varepsilon_r^p)^{H(p,q)}. \qquad (4)$$

Scaling of the form given in (2) clearly implies ESS.

Note that the ESS property does not simply reduce to a scaling property. It betrays an underlying multiplicative cascade mechanism and describes the evolution of the probability density functions of a scale dependent quantity (e.g., velocity increments or locally averaged dissipation in turbulence), denoted by ε_r here, from the larger scales to the finer. Indeed, the ESS can be seen as the signature of what is called an *infinitely divisible cascade scaling*. This was first observed on 1D signals such as hot wire velocity measurements in turbulence [11,20,35,46] or Internet traffic flows [42,48]. The framework of *infinitely divisible scaling* [10,13] allows for more flexible scaling and thus better fitting of data and honours the contribution of all scales in a range of interesting scales $r_{min} \leq r \leq r_{max}$ as follows:

$$\mathbb{E}\varepsilon_r^q = C_q \exp[-\tau(q)n(r)],\ r_{min} \leq r \leq r_{max}, \qquad (5)$$

where $n(r)$ is some monotonous function. In terms of scale dependence, the infinitely divisible scaling framework generalizes (2) which is recovered by choosing $n(r) = -\ln r$. The difference in spirit lies in the fact that multifractal analysis applies to any process and is concerned with local properties in the limit of fine scales, but not finite scales. Note that both multifractal analysis and infinitely divisible scaling can be formulated using wavelet coefficients [5,25,26,48]. Moreover, the infinitely divisible scaling approach is deeply connected to multiplicative cascades and appeared as a good entry to infinitely divisible cascades.

Beyond statistical analysis, there is also a need for actual models and tools to *synthesize* processes with controllable scaling properties. To this respect, multiplicative cascades appear intimately connected to multifractal processes so that they have played a key role in turbulence. A nice feature of multiplicative cascades is that their synthesis relies on an easy to implement iterative procedure. A succession of refinements and generalizations of such multiplicative cascades led to the *infinitely divisible cascades* (IDC). IDC are a versatile family of non Gaussian scale invariant processes which are easy to synthesize numerically. For 1D signals (time series), IDC have given a way to the synthesis of a large family of multifractal processes with prescribed properties [7,34,16–18]. This paper aims at showing how the construction of infinitely divisible cascades in 1 dimension generalizes to d dimensions, $d \geq 2$, with again many appealing properties: scaling exponents can be prescribed in (2); the scaling range can be precisely defined; properties are observed over a continuum in space and scale, e.g., there is no preferred scale ratio as in discrete constructions; a wide class of non Gaussian models is available. In d dimensions, geometrical features (e.g., anisotropy) can be interestingly taken into account. Such properties are very useful to build a relevant model of turbulent data (see Sect. 4) or to model natural images and textures as explained in a forthcoming paper [12].

The article is organized as follows. In Section 2, we extend the definitions of (scalar) infinitely divisible cascades (IDC) from 1 to d ($d \geq 2$) dimensions and review their main properties. In Section 3, we focus on the vast subclass of Compound Poisson Cascades (CPC) which turn to be easy to synthesize numerically and receive interesting physical interpretation. In Section 4, we propose a phenomenology of the intermittency phenomenon in fully developed turbulence viewed through the lens of infinitely divisible cascades to finally review some classical models of turbulence within this framework.

2 Infinitely divisible cascades in d dimensions

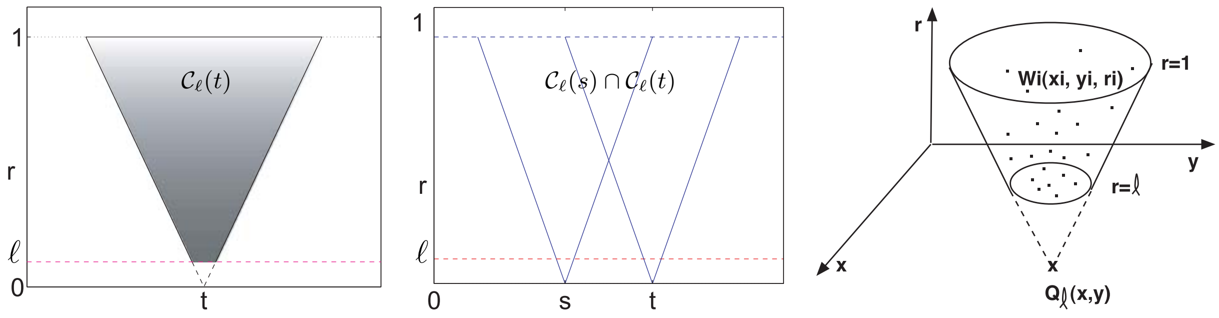
Recently, IDCs [7,34,4,16–18,44] have been introduced in 1 dimension as a “randomized version” of the well known canonical multiplicative cascades of Mandelbrot [32,49]. Mandelbrot’s canonical cascades, also called binomial cascades, are built on a dyadic tree from the larger to the smaller scales. Despite many interesting properties, these cascades have two main drawbacks. Because of the dyadic structure, binomial cascades display *discrete scale invariance* only. Moreover, such a construction is not *invariant to translation* so that the resulting process is *not stationary* in the strict sense. IDCs provide us with a versatile tool of stochastic modelling since they achieve *continuous scale invariance* as well as *true stationarity*. Some features are easier to describe in 1 dimension while they extend to d dimensions quite naturally as will appear in the sequel. Therefore, we first briefly recall the definition of an IDC $Q_\ell(t)$ in 1 dimension [7,34,4,16–18,44]. Then we define IDCs $Q_\ell(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^d$ in higher dimension $d \geq 2$.

2.1 Definitions

Let G be an infinitely divisible distribution with moment generating function $\tilde{G}(q)$ that can be written in the form $e^{-\rho(q)}$.

Let $dm(t,r) = g(r)dtdr$ a positive measure on the time-scale half-plane $\mathcal{P}^+ := \mathbb{R} \times \mathbb{R}^+$.

Let M denote an infinitely divisible, additive independently scattered random measure distributed by G, and supported on the time-scale half-plane $\mathcal{P}^+$ and associated to its so-called control measure $dm(t,r)$. For all disjoints subsets $\mathcal{E}_1$ and $\mathcal{E}_2$, $M(\mathcal{E}_1)$ and $M(\mathcal{E}_2)$ are independent random variables and $M(\mathcal{E}_1 \cup \mathcal{E}_2) = M(\mathcal{E}_1) + M(\mathcal{E}_2)$. The random measure M is such that $\mathbb{E}[\exp[qM(\mathcal{E})]] = \exp[-\rho(q)m(\mathcal{E})]$.



(a) (b) (c)

Fig. 1. "Time-scale" construction of Infinitely Divisible Cascades. (a) The shaded cone indicates the region that determines the value of the cascade at time t. (b) The dependence between $Q_\ell(t)$ and $Q_r(s)$, in particular their correlation, stems entirely from the measure of the intersection of two cones $\mathcal{C}_r(t)$ and $\mathcal{C}_\ell(s)$. (c) Space-scale cone defining $Q_\ell(\mathbf{x})$ at $\mathbf{x}(x, y)$. For a compound Poisson cascade, Q_ℓ is the product of those random multipliers $W_i(x_i, y_i, r_i)$ that belong to the cone $\mathcal{C}_\ell(\mathbf{x})$.

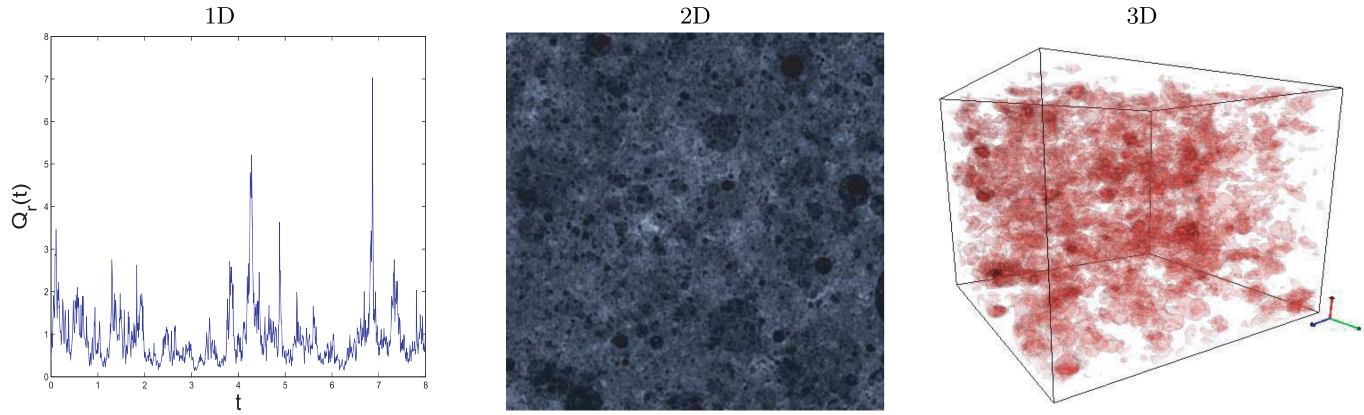


Fig. 2. Examples of an IDC in 1, 2 and 3 dimensions respectively.

Definition (in 1 dimension)
For given resolution $0 < \ell \leq 1$, let $\mathcal{C}_\ell(t)$ the *cone of influence*[2] defined for every $t \in \mathbb{R}$ as $\mathcal{C}_\ell(t) = \{(t', r') : \ell \leq r' \leq 1, t - r'/2 \leq t' \leq t + r'/2\}$ (see Fig. 1a). An *Infinitely Divisible Cascading noise* (IDC noise) is a family of processes $Q_\ell(t)$ parametrized by ℓ of the form

$$Q_\ell(t) = \frac{\exp\left[M(\mathcal{C}_\ell(t))\right]}{\mathbb{E}[\exp M(\mathcal{C}_\ell(t))]}. \tag{6}$$

Possible choices for distribution G are the Normal distribution, Poisson distribution, compound Poisson distributions, Gamma laws, Stable laws ... so that a large variety of choices is available for modelling and applications.

A nice property of IDC noises lies in the geometrical interpretation of their correlations that are controlled by the intersections of cones $\mathcal{C}_\ell(t) \cap \mathcal{C}_\ell(s)$ in the time-scale plane $\mathcal{P}^+$ — see Figure 1b. This is due to the properties of independence and additivity of the random measure M. This degree of freedom has been explored in [17,18] to get *warped infinitely divisible cascades* which display a controlled departure from power law scaling behaviours.

[2] Note that the large scale in the definition of $\mathcal{C}_\ell(t)$ has been arbitrarily set to 1 without loss of generality. Choosing a different large scale L would simply reduce to a change of units $t \to t \cdot L$, $\ell \to r \cdot L$.

The d-dimensional version is a natural generalization of the one-dimensional definition by simply extending the ingredients from 1 to d dimensions. The interest of this generalization will appear manifold: many more degrees of freedom are available in dimension $d \geq 2$ compared to dimension 1 and the range of potential applications is obviously much wider.

G is still an infinitely divisible distribution with moment generating function $\tilde{G}(q) = e^{-\rho(q)}$. The random measure M denotes an infinitely divisible, independently scattered additive random measure distributed by G, supported on the space-scale half-plane $\mathcal{P}^+ := \mathbb{R}^d \times \mathbb{R}^+$ and associated to its *control measure* $dm(\mathbf{x}, r) = g(r) d\mathbf{x} dr$.

Definition 1 (in d dimensions)
A *cone of influence* $\mathcal{C}_\ell(\mathbf{x})$ is defined for every $\mathbf{x} \in \mathbb{R}^d$ as $\mathcal{C}_\ell(\mathbf{x}) = \{(\mathbf{x}', r') : \ell \leq r' \leq 1, \|\mathbf{x}' - \mathbf{x}\| < r'/2\}$ — see Figure 1c. With a given infinitely divisible randomly scattered measure M, an *Infinitely Divisible Cascading noise* (IDC noise) is a family of processes $Q_\ell(\mathbf{x})$ parametrized by $\ell \in (0, 1)$ of the form (see Fig. 2 for examples in 1D,

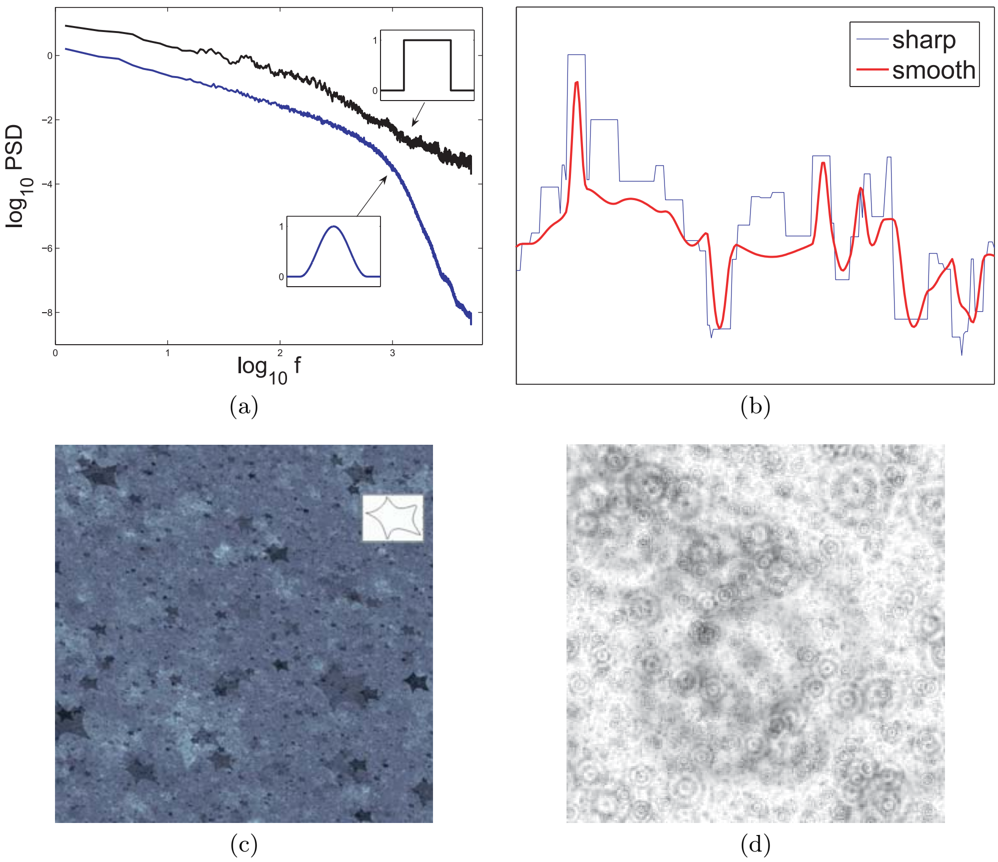


Fig. 3. (a) Smoothing effect of the kernel: comparison between the spectra of two different Q_ℓ obtained in 1 dimension from two different shapers, $f_1(x) = \mathbb{1}_{\{|x|<1/2\}}$ and $f_2(x) = \cos^2(|x|) \cdot \mathbb{1}_{\{|x|<1/2\}}$. (b) Realizations obtained using the same random variables $\{W_i, x_i, y_i, r_i\}$. The use of a regular kernel f implies a smoother (small scales) behaviour of Q_ℓ. (c) and (d) Examples of IDC generated from some specific kernels: the geometry of the kernel influences the local structure of the field (here 2D textures).

2D and 3D):

$$Q_\ell(\mathbf{x}) = \frac{\exp M(\mathcal{C}_\ell(\mathbf{x}))}{\mathbb{E}[\exp M(\mathcal{C}_\ell(\mathbf{x}))]}. \tag{7}$$

where $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$.

The previous definition may be extended to an even more general framework by introducing some localized integration kernel[3] $f(\mathbf{x})$ in (7):

Definition 2 (with integration kernel).

$$Q_\ell(\mathbf{x}) = \frac{\exp \int_{\ell \le r' \le 1} f(\frac{\mathbf{x}-\mathbf{x}'}{r'})\, dM(\mathbf{x}', r')}{\mathbb{E}\left[\exp \int_{\ell \le r' \le 1} f(\frac{\mathbf{x}-\mathbf{x}'}{r'})\, dM(\mathbf{x}', r')\right]}. \tag{8}$$

This definition may be useful for various purposes. It reduces to definition 1 when f is simply the indicating function $\mathbb{1}_\mathcal{D}$ of the disk of radius 1/2 $\mathcal{D} = \{\mathbf{x} : \|\mathbf{x}\| < 1/2\}$. In the following, we refer to this default kernel as the *cylindrical kernel*. This kernel displays sharp edges that generate sharp variations at small scales in $Q_\ell(\mathbf{x})$. One may obtain a field Q_ℓ that is smoother at small scales by choosing a smoother function f that will act as a regularization kernel. Indeed, choosing a smooth function f attenuates the small scales discontinuities — see Figures 3a and 3b. Furthermore, the choice of f permits to take into account some geometrical features (e.g., anisotropy) of a multifractal scalar field to be modelled — see Figure 3c where a non circular five branch star cone has been used. This degree of freedom may be very useful as far as applications such as the modelling of anisotropic flows or texture synthesis are concerned. For instance, to an even more general extent, one may consider the use of a kernel evolving from anisotropic at large scales to isotropic at small scales to take into account the decay of anisotropy in turbulent flows at fine scales. Theoretical difficulty will then rise from the lack of control on the properties of the resulting process. There remain several interesting open questions.

Aiming at smooth variations, one may use as regular as desired functions $f(\mathbf{x})$ with compact support (e.g., the disk $\mathcal{D} = \{\mathbf{x} : \|\mathbf{x}\| < 1/2\}$) and a maximum equal to one. Thus, $f(\mathbf{x})$ shall look as a unit height pulse with smooth edges, like a squared-cosine bell for instance, $f(\mathbf{x}) = \cos^2(\pi\mathbf{x})$ for $||\mathbf{x}|| \le 1/2$. The function f may even not have a compact support. It should then have sufficiently fast decreasing tails, like a Gaussian bell of unit

[3] This may be related to the *random wavelet expansions* proposed in [33].

width for instance. Finally, one may even choose some oscillating function, e.g., for aesthetic purpose — see Figure 3d. Some sophisticated mathematical work is still left to consider all possible choices of function $f(\mathbf{x})$.

2.2 Basic properties

2.2.1 Immediate consequences of the definition

Q_ℓ is a stationary positive random process with:

$$\mathbb{E}Q_\ell = 1. \tag{9}$$

Stationarity is ensured by the specific choice of a translation invariant *control measure* $dm(\mathbf{x}, r) = g(r)d\mathbf{x}dr$ and a translation invariant cone $\mathcal{C}_\ell(\mathbf{x})$.

The distribution of $Q_\ell(\mathbf{x})$ is log-infinitely divisible [21]. This is simply because $\log Q_\ell$ is an infinitely divisible random variable with distribution G associated to the stochastic measure M. Lots of distributions can be used since many of the distributions having a known explicit density (Gaussian, Poisson, Gamma...) are infinitely divisible [21]. Thus, we get a versatile family of non Gaussian scale invariant models to describe scalar fields in d dimensions with non trivial correlations. Section 4 will show how these processes may serve to model turbulent flows.

2.2.2 The non degeneracy criterion

At this point, we should mention that from a purely mathematical viewpoint, the right object to look at is not Q_ℓ itself. Indeed, without entering into too much details, for given resolution ℓ, Q_ℓ is the density of a measure A_ℓ such that for any compact set $\mathcal{E} \subset \mathbb{R}^d$,

$$A_\ell(\mathcal{E}) = \int_{\mathcal{E}} Q_\ell(\mathbf{x}')\, d\mathbf{x}'. \tag{10}$$

Taking the limit $\ell \to 0$, one gets the limiting measure A such that:

$$A(\mathcal{E}) = \lim_{\ell \to 0} \left[\int_{\mathcal{E}} Q_\ell(\mathbf{x}')\, d\mathbf{x}' \right]. \tag{11}$$

Thus some non degeneracy criterion has to be verified by the chosen distribution G [27,7] so that A be well defined, i.e. $\rho'(1) - \rho(1) > -d$, otherwise A degenerates to zero.

The limit $\ell \to 0$ is not reached in numerical simulations but this non degeneracy criterion must be taken care of. If not, the simulated Q_ℓ may display a very special behaviour with very high isolated peaks among an ocean of nearly zero values. In certain cases, this may happen even when ℓ is not very small (e.g., $\ell \sim 0.01$). We emphasize that this non degeneracy criterion which could appear at first sight as an abstract mathematical property is of crucial practical importance.

2.2.3 Scaling properties

First, we focus on the more simple definition (7) (with cylindrical kernel). In this case, the measure M, the distribution G, the control measure m and the geometry of the cone of influence $\mathcal{C}_\ell(\mathbf{x})$ control the scaling structure as well as marginal distributions of the cascade. One major property of IDCs is:

$$\mathbb{E}[Q_\ell^q] = \exp\left[-\varphi(q)\, m(\mathcal{C}_\ell)\right] \tag{12}$$

where

$$\varphi(q) = \rho(q) - q\rho(1), \qquad \varphi(1) = 0, \tag{13}$$

for all q for which $\rho(q) = -\log \tilde{G}(q)$ is defined. Note the similarity between (12) and (5).

Turning to local averages ε_r over a volume V_r, the mathematician would look at the moments of the limiting measure A defined in previous section. As already mentioned, numerical simulations only give access to the measure A_ℓ with finite resolution. In this spirit we prefer to describe the scaling properties of the practical quantity

$$\varepsilon_r(\mathbf{x}) = \frac{1}{V_r} \int_{\|\mathbf{x}'-\mathbf{x}\|<r} Q_\ell(\mathbf{x}')\, d\mathbf{x}' = \frac{1}{V_r} A_\ell(V_r), \tag{14}$$

even though they are deduced from those of the "more mathematical" quantity $A(V_r)$.

The scaling properties of IDCs in 1 dimension have been studied in [4,6,7,17,18,34,44]. They can be extended naturally to IDC in d dimensions:

$$\mathbb{E}\varepsilon_r(\mathbf{x})^q \propto \exp\left[-\tau(q)\, m(\mathcal{C}_r)\right] \tag{15}$$

where in general $\tau(q) = \varphi(q)$ at least within some limited range of values of q.

Power law scaling behaviours are intimately connected to the particular choice of the control measure [4,17,18,34] (see Appendix A):

$$dm(\mathbf{x}, r) = \frac{dr}{V_{1/2} r^{d+1}} d\mathbf{x}, \qquad 0 < r \le 1, \tag{16}$$

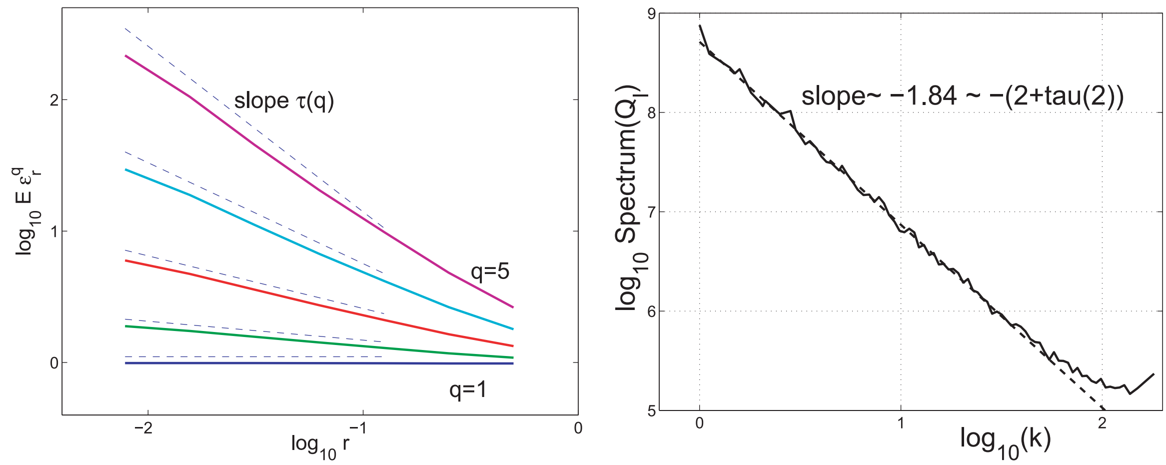
where $V_{1/2}$ is the volume of the sphere of radius $1/2$ in d dimensions (e.g., $\pi/4$ in 2D). Then $m(\mathcal{C}_r) = -\log r$ so that[4]:

$$\mathbb{E}\varepsilon_r(\mathbf{x})^q \sim r^{\tau(q)} \text{ for } r \ll 1. \tag{17}$$

See Figure 4b for an example of numerical results on cascades in 2D. In practice, estimates of $\tau(q)$ identify to $\varphi(q)$ within a finite range of values of $q \in (q_-^*,\ q_+^*)$ only. The precise values of the lower and upper bounds q_-^* and q_+^* are determined by solving $\varphi(q) - q\varphi'(q) = N$ (see [31] and references therein). As a consequence, a data analysis cannot identify the function $\varphi(q)$ for any order q but only in the finite range $q \in (q_-^*, q_+^*) \cup [0, 1]$.

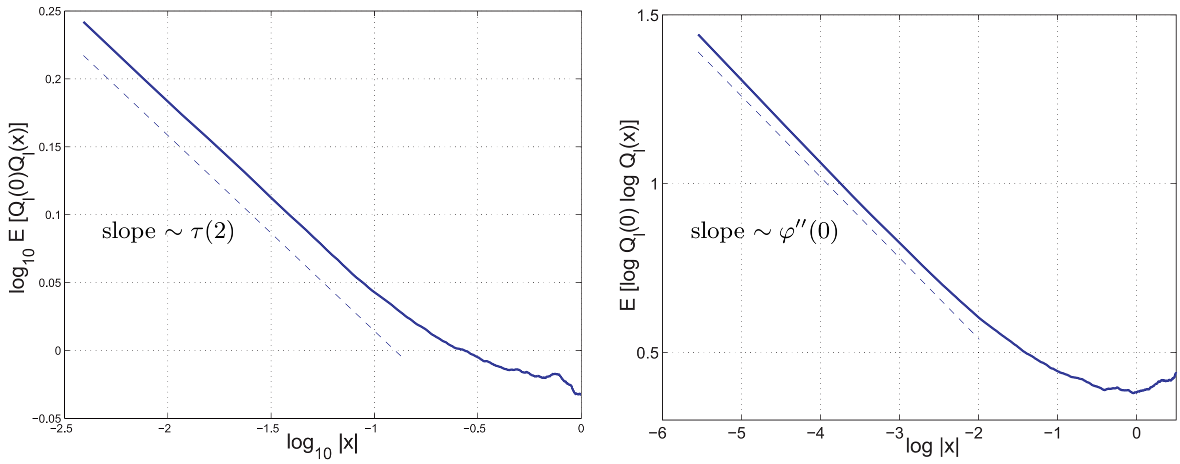
Another usual evidence for a power law scaling behaviour is the observation of a power law spectrum proportional to $1/k^\alpha$. Infinitely divisible cascades display a

[4] In 1 dimension, the choice $dm(t, r) = \left(\frac{\mathbb{I}_{(0,1]}}{r^2} + \delta(1-r)\right) dr\, dt$ yields exact power laws [17,34].



(a) (b)

Fig. 4. (a) Box averages of a 2D IDC obey scaling laws of the form given by (17). (b) Power law spectrum of $Q_\ell(\mathbf{x})$ in 2 dimensions as a function of $k = \|\mathbf{k}\|$ over 2 decades: the observed slope is prescribed by the choice of $\tau(2) = \varphi(2)$ here.



(a) (b)

Fig. 5. Estimate of correlation functions on 2D cascades. (a) $\mathbb{E}[\mathbf{Q}_\ell(\mathbf{0})\mathbf{Q}_\ell(\mathbf{x})] \propto |\mathbf{x}|^{\tau(\mathbf{2})}$ over nearly 2 decades: the observed slope exponent is prescribed by the choice of $\tau(2) = \varphi(2)$ here. (b) $\mathbb{E}[\log \mathbf{Q}_\ell(\mathbf{0}) \log \mathbf{Q}_\ell(\mathbf{x})] \propto \log |\mathbf{x}|$ betrays a multiplicative correlation structure. The slope is prescribed by $\varphi''(0)$ sometimes called the intermittency coefficient.

power law spectrum of the form $1/k^{d+\tau(2)}$ ($\tau(2) < 0$) as illustrated in Figure 4a in a 2D case (averaging over all directions).

The scaling properties of the most general IDCs as defined by (8) take the same form except that $\varphi(q)$ is replaced by (if the integral converges, see Proposition 2.6 in [39]):

$$\tau(q) = \int \varphi(q f(\mathbf{x})) d\mathbf{x}. \tag{18}$$

Therefore, the control of the scaling behaviour is much more difficult since $\tau(q)$ combines information both from $\varphi(q)$ and from the kernel function $f(\mathbf{x})$ in a non linear way. One may control and prescribe the value of $\tau(q)$, in numerical simulation at least, by using successive approximations. For sake of clarity, scaling exponents will always be denoted by $\tau(q)$ (or $\zeta(q)$ for velocity increments in turbulence) in the sequel.

Turning to autocorrelation functions, one expects [4,18,34] at scales smaller than 1 (1 is the largest scale where the cascade begins):

$$\mathbb{E}[Q_\ell(0) Q_\ell(\mathbf{x})] \propto |\mathbf{x}|^{\tau(2)} \text{ for } |\mathbf{x}| \ll 1 \tag{19}$$

which can be checked numerically, see Figure 5a. As a consequence of the multiplicative construction, one expects as well that (see Appendix B):

$$\mathrm{cov}(\log Q_\ell(0), \log Q_\ell(\mathbf{x})) \propto \varphi''(0) \log |\mathbf{x}| \text{ for } |\mathbf{x}| \ll 1. \tag{20}$$

Indeed, the amount of information common to $\log Q_\ell(0)$ and $\log Q_\ell(\mathbf{x})$ can be intuitively and quantitatively related to the number of ancestors common to $Q_\ell(0)$ and $Q_\ell(\mathbf{x})$ which is by construction proportional to $\log |\mathbf{x}|$. This behaviour is illustrated in Figure 5b. Note that this logarithmic behaviour is also consistent with the observation of a power-law spectrum for the process $\log Q_\ell(\mathbf{x})$. As a consequence, the scale invariance property (at least considered as the presence of a power law spectrum) can be observed both on Q_ℓ and $\log Q_\ell$. This remark may reveal of particular interest when dealing with the statistical modelling of natural images, see, e.g., [12,47]. Indeed, authors are sometimes confusing about whereas they observe scale invariance on the intensity $I(\mathbf{x})$ of the image or its logarithm $\log I(\mathbf{x})$. With little paradox, both quantities may display a power law spectrum [12].

3 A special family: compound Poisson cascades

We have now the most general definitions and properties of IDC at our disposal. In practice, not all IDC can be easily synthesized. Compound Poisson Cascades (CPC) are a very interesting subclass of IDCs that are very easy to synthesize numerically even in d dimensions. This section aims at gathering all useful definitions and properties of CPC as well as the description of an algorithm for numerical synthesis in d dimensions.

3.1 Definitions

They were first introduced in 1 dimension in the seminal work by Barral & Mandelbrot [7] as *Multifractal Products of Cylindrical Pulses* (in one dimension only). It appears that they relate to classical models proposed to describe the statistics of both turbulent flows [20,45,46] — see Section 4 — and natural images [12,19,24,33]. Indeed, compound Poisson cascades reveal as the most useful family of IDCs for applications. The key idea was to replace the dyadic tree structure $\{(t_{j,k}, r_{j,k}) = ((k-1/2)2^{-j}, 2^{-j}),\ j \in \mathbb{N},\ k \in \mathbb{Z}\}$ of binomial cascades by a well chosen *random Poisson point process* (t_i, r_i) in the time-scale plane, see Figure 1c. Aiming at power law scaling in 1 dimension, "well chosen" means that it has density $dm(t,r) = dtdr/r^2$ — see (16). Thus, the density of points increases as $r \to 0$ exactly as it increases in a dyadic grid. Note that this density is time-shift invariant. Therefore, MPCP are *stationary.* Moreover, scaling laws are observed over a continuous range of scales since no privileged scale ratio has been introduced. Random i.i.d. positive multipliers W_i are associated to vertices (t_i, r_i). Then a compound Poisson cascade is defined in 1D by:

$$Q_\ell(t) = \frac{\prod_{(t_i,r_i)\in\mathcal{C}_\ell(t)} W_i}{\mathbb{E}\left[\prod_{(t_i,r_i)\in\mathcal{C}_\ell(t)} W_i\right]}. \qquad (21)$$

Again, this definition naturally extends to d dimensions by replacing $t_i \in \mathbb{R}$ by $\mathbf{x}_i \in \mathbb{R}^d$, the 2D cone $\mathcal{C}_\ell(t)$ by a $(d+1)$-dimensional cone $\mathcal{C}_\ell(\mathbf{x})$. Power law scaling behaviours are recovered by choosing $dm(\mathbf{x}, r) = d\mathbf{x}dr/r^{d+1}$ (see Sect. 2.2.3 and Appendix A):

$$Q_\ell(\mathbf{x}) = \frac{\prod_{(\mathbf{x}_i,r_i)\in\mathcal{C}_\ell(\mathbf{x})} W_i}{\mathbb{E}\left[\prod_{(\mathbf{x}_i,r_i)\in\mathcal{C}_\ell(\mathbf{x})} W_i\right]}. \qquad (22)$$

Taking the logarithm one gets:

$$\log Q_\ell(\mathbf{x}) = \sum_{(\mathbf{x}_i,r_i)\in\mathcal{C}_\ell(\mathbf{x})} \log W_i + K, \qquad (23)$$

where K is a normalisation constant. Equation (23) can be seen as the random measure $M(\mathcal{C}_\ell(\mathbf{x}))$ of the cone $\mathcal{C}_\ell(\mathbf{x})$. Definition (22) appears as a particular form of (7) where the random measure M is built as a sum of Dirac pulses of random weights $\log W_i$ located at random positions $(\mathbf{x}_i, r_i)$. This remark makes clear that MPCP may be called *Compound Poisson Cascades* since the distribution of $\log Q_\ell(t)$ is a compound Poisson distribution. It appears that the Poisson distribution of the point process $(\mathbf{x}_i, r_i)$ is compound with the distribution F of $\omega_i = \log W_i$.

One step further, let us note that (23) can be written in the equivalent form:

$$\log Q_\ell(\mathbf{x}) = \sum_i \log W_i \, \mathbb{1}_{\mathcal{D}(\mathbf{x}_i,r_i)}(\mathbf{x}) + K, \quad (K = \text{constant}), \qquad (24)$$

where $\mathbb{1}_{\mathcal{D}(\mathbf{x}_i,r_i)}$ is the indicating function of the disk of centre $\mathbf{x}_i$ and diameter r_i. This suggests to replace $\mathbb{1}_{\mathcal{D}(x_i,r_i)}$ by some function $f((\mathbf{x}-\mathbf{x}_i)/r_i)$ which yields the general form:

$$\log Q_\ell(\mathbf{x}) = \sum_i \log W_i \cdot f\left(\frac{\mathbf{x}-\mathbf{x}_i}{r_i}\right) + K. \qquad (25)$$

One may see $f(\mathbf{x})$ as the geometrical descriptor of some generic ingredient in the cascade. For instance, it may be related to the geometry of dissipative structures in turbulent flows (see Sect. 4) or of objects in images (see [12,24]). Taking the exponential of (25), we are back to the equivalent formulation of (8) for compound Poisson cascades:

$$Q_\ell(\mathbf{x}) = \frac{\prod_i W_i^{f\left(\frac{\mathbf{x}-\mathbf{x}_i}{r_i}\right)}}{\mathbb{E}\left[\prod_i W_i^{f\left(\frac{\mathbf{x}-\mathbf{x}_i}{r_i}\right)}\right]}. \qquad (26)$$

Another possible extension of (22) was proposed and studied in [6] (in one dimension only) of the form:

$$\tilde{Q}_\ell(\mathbf{x}) = \frac{\prod_{(\mathbf{x}_i,r_i)\in\mathcal{C}_\ell(\mathbf{x})} \tilde{P}_i(\mathbf{x})}{\mathbb{E}\left[\prod_{(\mathbf{x}_i,r_i)\in\mathcal{C}_\ell(\mathbf{x})} \tilde{P}_i(\mathbf{x})\right]} \qquad (27)$$

where $\tilde{P}_i(\mathbf{x}) = W_i \tilde{W}(\frac{\mathbf{x}-\mathbf{x}_i}{r_i}) 1_{\mathcal{D}(\mathbf{x}_i,r_i)} + 1_{\complement\mathcal{D}(\mathbf{x}_i,r_i)}$; $\mathcal{D}(\mathbf{x}_i, r_i)$ denotes the disk of R^d of centre $\mathbf{x}_i$ and radius r_i, and $\tilde{W}(\mathbf{x})$ is some non-negative function in $L^1(\mathcal{D}(0,1/2))$ such that $\int_{\mathcal{D}(0,1/2)} \tilde{W} = 1$. While such an approach may appear natural as far as compound Poisson cascades are concerned, it is difficult to generalize to infinitely divisible cascades in general. This is the reason why we rather chose to use (8) and (26) in place of (27).

3.2 Interpretation and properties

We will mainly focus on compound Poisson cascades not only because their synthesis is simple to implement in

d dimensions (see Sect. 3.4) but also because they have an intuitive interpretation. Notably, they have been often evoked in the context of turbulence [9,13,20,43,46] — see Section 4.4.3. This is mainly due to the usual easy interpretation of compound Poisson distributions that we briefly recall below.

Let $\Delta n \in \mathbb{R}^+$ and let F be the probability density function of some random variable ω. The probability density function of a compound Poisson distribution is defined by

$$G_{\Delta n} = e^{-\Delta n} \sum_{k=1}^{\infty} \frac{\Delta n^k}{k!} F^{*k} \tag{28}$$

where F^{*k} is the kth convolution of F. A classical interpretation of this definition rises by considering the following process. Let $N(\Delta n)$ be a Poisson random variable of expectation Δn and distributed by

$$\mathcal{P}_{\Delta n}\{N(\Delta n) = k\} = e^{-\Delta n} \frac{\Delta n^k}{k!}. \tag{29}$$

For each value of k, let $\omega_1, \omega_2, \ldots, \omega_k$ be k independent random variables distributed by F, independent of k. Then (28) describes the probability density function of the random sum $x_N = \omega_1 + \omega_2 + \ldots + \omega_{N(\Delta n)}$.

Now let $X_N = e^{x_N}$ and $W = e^{\omega}$. Then

$$X_N = W_1 \ldots W_{N(\Delta n)}.$$

It is easy to prove that $\mathbb{E}X_N^q = \mathbb{E}e^{qx_N} = \exp[-\rho(q) \cdot \Delta n]$ with $\rho(q) = 1 - \mathbb{E}W^q$. Let G the distribution defined by $\tilde{G}(q) = e^{-\rho(q)}$. Then (28) reads:

$$G_{\Delta n} = G^{*\Delta n}. \tag{30}$$

and one gets in (13):

$$\varphi(q) = 1 - \mathbb{E}[W^q] - q(1 - \mathbb{E}W). \tag{31}$$

3.3 An example in turbulence

Let $\underline{F}$ denote the probability density function of W with $F(\omega) = W\underline{F}(W)$. As a particular case of interest we consider the choice

$$F(\omega) = \lambda e^{\lambda\omega}, \quad \omega \in (-\infty,\ 0] \Leftrightarrow \underline{F}(W) = \lambda W^{\lambda-1}, \qquad \lambda > 0, W \in [0,1]. \tag{32}$$

The case $\lambda = 1$ corresponds to uniformly distributed variables W in $[0,1]$. This choice (32) was proposed by Ambarzumian [1] in his work on the Milky Way (see Sect. 4.3) but also implicitly appears in works by Castaing [9] and Yakhot [50] in turbulence. A very similar distribution was obtained independently by Grenander & Srivastava [24] as well to describe the statistics of natural images (see also [12]). The kth convolution F^{*k} of F is then given by:

$$F^{*k}(\omega) = \gamma_{k,\lambda}(|\omega|) = \frac{\lambda^k}{(k-1)!} |\omega|^{k-1}\, e^{-\lambda|\omega|}, \qquad -\infty < \omega \leq 0. \tag{33}$$

Reporting (33) in (28) we get a distribution $G_{\Delta n}(\omega)$ with an atom $e^{-\Delta n}$ at the origin ($\omega = 0 \Leftrightarrow W = 1$) and described for $\omega < 0$ by:

$$\begin{aligned} G_{\Delta n}(\omega) &= e^{-\Delta n} \sum_{k=1}^{\infty} \frac{(\lambda\Delta n)^k \mid \omega \mid^{k-1}}{k!(k-1)!} e^{-\lambda|\omega|} \\ &\Longleftrightarrow \\ G_{\Delta n}(\omega) &= e^{-\Delta n}\, \sqrt{\lambda\Delta n}\, \frac{e^{-\lambda|\omega|}}{\sqrt{|\omega|}}\, I_1(2\sqrt{\lambda\Delta n|\omega|}), \\ &\quad \text{with } \omega \in]-\infty,\ 0[, \end{aligned} \tag{34}$$

where I_1 is the modified Bessel function of order 1 [21]. $G_{\Delta n}$ in (34) is infinitely divisible since it is a compound Poisson distribution. In the work by Castaing [9], the parameter $T = 1/\lambda$ is called "temperature of a turbulent flow". Thus, compound Poisson distributions generate a very interesting family of IDCs to model turbulent flows (see Sect. 4) as well as natural images (see [24,12]). We emphasize that scalar CPC models such as the example presented above can be easily synthesized numerically as explained in the next section. The synthesis of vectorial (not only scalar) fields with similar properties is the subject of ongoing research.

3.4 Algorithm for synthesis

Algorithms for synthesis in 1 dimension have been described in great detail in [34,17] so that we simply mention below the general ideas and main difficulty when generalizing to the synthesis of an IDC in d dimensions for $d \geq 2$.

In the most general case, the synthesis of an IDC in d dimensions is a difficult question since it rises the problem of the simulation of an infinitely divisible random measure $dM(\mathbf{x}, r)$ in a space of dimension $d+1 \geq 3$... Even though it is not impossible, at least in principle, it calls for the description of complicated domains (intersections of cones in a (d+1)-dimensional space...).

Hopefully, the simulation of a CPC is easier since it only relies on the use of a marked Poisson point process. Thus the algorithm for synthesis works by construction with discrete sets of variables only. Moreover, the use of a shaper as in (8) is easy and makes this family of processes very versatile for data modelling or texture synthesis in d dimensions.

Let the trapezoid volume $\Theta = \{(\mathbf{x}', r') : \forall 1 \leq i \leq d, \ell \leq r' \leq 1, -r'/2 \leq x_i' \leq X_i^{max} + r'/2\}$. The main steps of the synthesis are for given resolution ℓ and size limitations $X_1^{max}, ..., X_d^{max}$:

1. determine the number N_p of points (and multipliers) that will be used to compute $Q_\ell(\mathbf{x})$ in the cubic domain $[0,\ X_1^{max}] \times ... \times [0,\ X_d^{max}]$: it is a Poisson random variable with parameter $m(\Theta)$;
2. select N_p random points $(\mathbf{x}_i, r_i)$ located in the trapezoid Θ, according to density $dm(\mathbf{x}, r)$[5];

[5] The non-uniform distribution $r^d g(r)$ of the r_i is achieved by a change of variable from a uniformly distributed random variable.

3. select N_p i.i.d. random multipliers W_i such that $\log W_i$ are distributed by F;
4. for each position $\mathbf{x} \in \{\mathbf{x}_k = (k_1 \Delta x, ..., k_d \Delta x)\ 0 \leq k_i \leq X_i^{max}/\Delta x\}$, set

$$Q_\ell(\mathbf{x}) = \exp[(1 - \mathbb{E}W) m(\mathcal{C}_\ell(\mathbf{x}))] \cdot \prod_{(\mathbf{x}_i, r_i) \in \mathcal{C}_\ell(\mathbf{x})} W_i.$$

A key feature of this algorithm is that it is rather easy to implement. A set of software are available from the author's web page at `www.isima.fr/ pchainai/PUB/software.html` to synthesize 1D, 2D and 3D fields in MATLAB (compiled C programs are available as well). A special software has been developed to visualize 3D densities.

We emphasize that one may as well consider $Q_\ell(\mathbf{x})$ in d dimensions as a dynamical field in d-1 dimensions by considering $\mathbf{x} = (x_1, ..., x_{d-1}, t)$. A "multifractal film" built following this principle can be downloaded from the author's web site. Thus, one can even consider the synthesis of a dynamical "turbulent" 3D scalar field from a compound Poisson cascade in 4 dimensions (3D + t). The differences between space variations and time variations can be taken into account using an anisotropic cascade. This might be interesting to model turbulent scalar fields (e.g., water or ice density in clouds) and is the subject of ongoing research.

4 IDC and the intermittency phenomenon in turbulence

The statistical modelling of turbulent flows rises a host of questions. Indeed, more than half a century has been devoted to the quest for a clear understanding of the statistics of fully developed turbulence. Despite some progress, there is still no fully deductive theoretical model including the so-called *intermittency phenomenon* in a 3D turbulent flow. We propose to model the cascade of energy in fully developed turbulence in the framework of compound Poisson cascades since such cascades receive a very appealing interpretation. Thanks to an analogy between the propagation of energy through the scales and the propagation of a ray of light through absorbing clouds, a versatile phenomenology of the cascade of energy is proposed. This phenomenology takes into account the intermittency phenomenon related to the non linear behaviour of scaling exponents $\tau(q)$. Moreover, previous works already focused on compound Poisson cascades [20,46] but were limited to a formal data analysis, while here we also provide a method for numerical synthesis of the model in 3 dimensions (even dynamical aspects may be injected as well by working in a 3D+t space).

4.1 Intermittency in turbulence

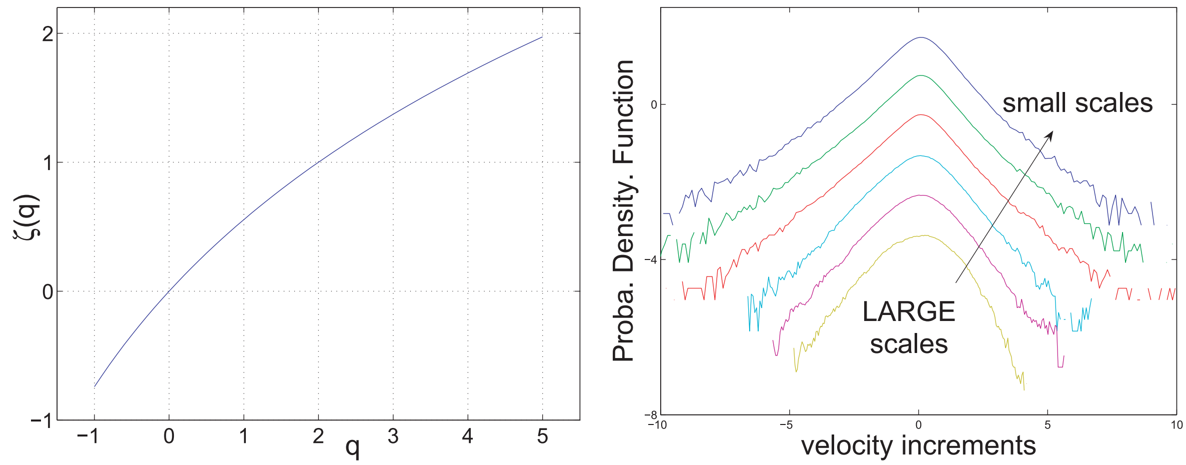
Intermittency in fully developed turbulence is commonly characterized by departures from the results of Kolmogorov 1941 theory [29]. The fundamental ideas underlying this theory are rooted in the phenomenology of the Richardson's cascade [40]. Richardson's cascade basically relies on the accommodation of two major ingredients: scale invariance and (dissipative) coherent structures in the flow. These two points are related to a *physical* qualitative interpretation of the behaviour of the non linear (quadratic) term of the Navier-Stokes equation that governs fluid flows: the energy is injected in the flow at the top of a hierarchy of eddies of decreasing size and is 'cascading' down to smaller and smaller eddies (because of the non linear term) to eventually dissipate entirely at the bottom of the hierarchy. In this view, no scale plays any specific role between the integral injection scale and the Kolmogorov dissipative scale: the main ingredient of Kolmogorov 1941 theory is self-similarity. In his 1941 paper [29], Kolmogorov predicted a self-similar scaling behaviour of the velocity structure functions of the form $\mathbb{E}|\delta v_r|^q \sim r^{q/3}$, where $\delta v_r = v(x+r) - v(x)$ is a longitudinal velocity difference over distance r. Two fundamental assumptions of this theory are (i) a constant and homogeneous rate of dissipation $\varepsilon(\mathbf{x}) = \varepsilon_o$; (ii) dissipation takes place at the dissipation scale only. Under such assumptions, the fractional Brownian motion with Hurst exponent $H = 1/3$ appears as a good model to describe the velocity field in turbulent flows (except for the non zero skewness of turbulent velocity fields). In summary, Kolmogorov 1941 theory can be presented as implicitly connected to the Gaussian self-similar stochastic process Kolmogorov had himself previously introduced from a mathematical point of view in [28]. The construction of a stochastic process with suitable scaling properties and a non zero skewness to model turbulent velocity fields is still an open question, even in 1 dimension.

In 1944, Landau objected that dissipation fluctuates so that self-similarity cannot be relevant [22]. Indeed, a large amount of experimental results later showed that the velocity increments rather behave as

$$\mathbb{E}|\delta v_r|^q \sim r^{\zeta(q)} \tag{35}$$

with $\zeta(q) \neq q/3$, which is usually referred to as the *intermittency phenomenon* or *anomalous scaling* property. In summary, the intermittency phenomenon is related to two connected empirical observations: the scaling exponents $\zeta(q)$ of the velocity increments structure functions in (35) do not behave linearly with q and the probability density functions of the velocity increments possess non Gaussian statistics, see Figure 6 using data from Baudet [8].

The *refined similarity hypothesis* proposed by Kolmogorov in 1962 [30] relates the non linear behaviour of exponents $\zeta(q)$ to the fluctuations of the locally averaged dissipation $\varepsilon_r(\mathbf{x})$ which Kolmogorov assumed to be log-Normal. In this second approach, Kolmogorov postulates that $\mathbb{E}\varepsilon_r^q \sim r^{\tau(q)}$ where $\tau(q)$ is a non linear function of q. The Karman-Howarth equation, a classical result deduced from Navier-Stokes equation, then suggests to postulate that $\mathbb{E}|\delta v_r|^q \sim r^{q/3+\tau(q)}$. This approach interestingly takes into account the non



(a) (b)

Fig. 6. (a) Scaling exponents $\zeta(q)$ of the velocity increments of a turbulent flow. (b) Evolution of the probability density functions of the increments from quasi Gaussian at large scales to non Gaussian at smaller scales.

Gaussian statistics of turbulent flows. Furthermore, it inspired the multifractal view of turbulence where data are seen as a collection of singularities and each singularity is more or less a signature of those cascading eddies. Historically, the reference processes for that situation are the multiplicative cascades which were introduced in the work by Yaglom [49], Novikov & Stewart [36,35] and Mandelbrot [32]. The purpose of the statistical modelling of turbulent flows has been and still is to accommodate scale invariance, statistical intermittency and coherent structures altogether within a consistent model. Without pretending to explain the intermittency phenomenon, we aim at proposing an original phenomenology of the cascade of energy in turbulence that generalizes Richardson's vision to a description that accounts for all of these three ingredients.

4.2 From infinitely divisible scaling to multiplicative cascades

In the 1990s, Castaing et al. [11,10] proposed the general framework of log *infinitely divisible scaling*. This approach makes explicit the fact that, beyond scaling laws, intermittency results in an evolution of the probability density functions of the velocity increments δv_r from Gaussian to non Gaussian shapes from large injection scales to small dissipative scales. Briefly, one remarks that

$$\mathbb{E}|\delta v_r|^q = \mathbb{E}e^{q\ln|\delta v_r|} = \mathbb{E}e^{qY_r} \tag{36}$$

is simply the moment generating function $\tilde{G}(q)$ of the variable $Y_r = \ln|\delta v_r|$. Thus, one may rewrite the scaling property (35) as a relative scaling:

$$\mathbb{E}|\delta v_{r_2}|^q \sim \left(\frac{r_2}{r_1}\right)^{\zeta(q)} \mathbb{E}|\delta v_{r_1}|^q. \tag{37}$$

Such a relative scaling behaviour enters the more general framework of the infinitely divisible scaling property:

$$\mathbb{E}|\delta v_{r_2}|^q \sim \exp[-\zeta(q)\Delta n(r_1,r_2)]\mathbb{E}[|\delta v_{r_1}|^q] \tag{38}$$

which is equivalent to the experimental observation of the so-called Extended Self Similarity (ESS) [3] — see (4). Using (36), this is equivalent to an equation that accounts for the evolution of the probability density functions of $Y_r = \ln|\delta v_r|$ from a large scale r_1 to a smaller one r_2 through the convolution operation:

$$P_{r_2}(\ln|\delta v_r|) = G^{*\Delta n(r_1,r_2)} * P_{r_1}(\ln|\delta v_r|), \tag{39}$$

where G is an infinitely divisible distribution with moment generating function $\tilde{G}(q) \propto e^{-\zeta(q)}$ [21]. Thus, a log infinitely divisible cascade is defined by its so-called propagator

$$G_{r_1,r_2} = G^{*\Delta n(r_1,r_2)}. \tag{40}$$

Power law scale invariance is recovered by setting $n(r) = -\ln r$ so that $\Delta n(r_1,r_2) = n(r_2) - n(r_1) = \ln(r_1/r_2)$ as in (37). The infinitely divisible distribution G is called the *kernel* and $\Delta n(r_1,r_2)$ is the depth of the cascade, i.e. the number of steps of the cascade from r_1 to r_2. Kolmogorov's 1941 and 1962 theories can be viewed as special cases where G is respectively a Dirac and a Normal distribution.

The fundamental ingredient of infinitely divisible scaling is the separation of variables q and r in the product $\zeta(q)\cdot\Delta n(r_1,r_2)$. Reading G as the distribution of a random variable $\tilde{\omega} = \ln\widetilde{W}$ independent of $Y = \ln|\delta v_r|$, we get:

$$\begin{aligned} P_{r_2}(Y) &= G_{r_1,r_2} * P_{r_1}(Y) \\ \Rightarrow Y(r_2) &\overset{d}{=} \tilde{\omega}_{r_1,r_2} + Y(r_1) \\ \Rightarrow \ln|\delta v_{r_2}| &\overset{d}{=} \ln\widetilde{W}_{r_1,r_2} + \ln|\delta v_{r_1}| \\ \Rightarrow |\delta v_{r_2}| &\overset{d}{=} \widetilde{W}_{r_1,r_2} \cdot |\delta v_{r_1}| \end{aligned} \tag{41}$$

where $\overset{d}{=}$ stands for equality of distributions. This equality betrays an underlying multiplicative structure of the process v. The evolution of the distributions of Y through the scales takes place "as if" there really was a multiplicative cascade:

$$\text{“}\quad |\delta v_{r_2}| \overset{d}{=} \widetilde{W}^{\Delta n(r_1,r_2)} \cdot |\delta v_{r_1}| \quad\text{”}. \tag{42}$$

Note that the variable $\widetilde{W}$ has no concrete meaning in this formulation. When (41) is obeyed by distributions $P_r(Y_r)$ there is no evidence that we can define some real process ("real" here means "having a physical existence" or "that can be simulated") $W_{r_1,r_2}(\mathbf{x}_1,\mathbf{x}_2)$ that associates $Y_{r_1}(\mathbf{x}_1)$ and $Y_{r_2}(\mathbf{x}_2)$ and such that G_{r_1,r_2} be the distribution of $\ln W_{r_1,r_2}$. We emphasize that this is a *descriptive* approach only. Moreover, Kolmogorov's *refined similarity hypothesis* [30] is necessary to link the properties of the velocity increments δv_r (associated to a vectorial field) to those of the locally averaged dissipation ε_r (associated to a scalar field).

However, such an approach to data analysis in turbulence played a crucial role in the elaboration of IDC. It actually makes an explicit connection between non-linear scaling exponents, departures from Gaussianity (statistical intermittency), infinitely divisible distributions and multiplicative cascades altogether. The understanding of infinitely divisible cascades introduced in [4,14,16,15,34, 44] as a model for process synthesis was directly inspired from this approach. Note that infinitely divisible scaling was used and developed while there was no model for synthesis at hand. It is remarkable that infinitely divisible cascades came later as the natural answer to an "old" question. By construction, IDC obey properties similar to (42) — e.g., see equation (40) in [34].

4.3 Ambarzumian's model: an analogy with the turbulent cascade

Let us consider a problem that has a priori no connection to turbulence and focus on a model proposed by Ambarzumian [1] in 1944 in a paper entitled *On the theory of brightness fluctuations in the Milky Way.* Eventually, we develop an analogy between the propagation of a ray of light through a distribution of absorbing clouds and the cascade of energy through the scales in turbulence. This analogy aims at clarifying the physical meaning of the various ingredients of the definition of IDC.

4.3.1 Ambarzumian's model of the Milky Way

In Ambarzumian's model, the variable r stands for a distance and $E(r)$ denotes the energy of a ray of light propagating through the space. Assume that every elementary volume radiates with constant rate so that $E(r)$ increases linearly ($\propto cr$). Assume moreover that space is filled with absorbing clouds that are distributed in a homogeneous and isotropic manner. Let $\rho = \alpha/\sigma$ be the number of clouds by unit volume; σ is an effective cross section and α is the average number of clouds by unit length of a straight line. When it gets through a cloud, a ray of light keeps only part of its energy due to partial transparency $W \in [0,\ 1]$ of the cloud. Let us denote by $F(\omega)$ and $\underline{F}(W)$ the respective probability densities of $\omega = \ln W$ and W. On one hand, $E(r)$ increases over distance r because of the homogeneous and isotropic radiation cr. On the other hand, random fluctuations are superimposed to this linear increasing trend due to absorption by clouds. These fluctuations are of main interest. Since clouds are uniformly distributed in space, the ray crosses k absorbing clouds with k distributed by a Poisson law of parameter αr: $\mathcal{P}_{\alpha r}(k) = e^{-\alpha r}(\alpha r)^k/k!$. Consequently, the ray of light undergoes a series of k attenuations by a factor W_i, $1 \le i \le k$, identically distributed by $\underline{F}$. The total attenuation is given by $W = W_1 \cdot \ldots \cdot W_k$. Taking logarithms, one gets the logarithmic attenuation $\omega = \omega_1 + \ldots + \omega_k$ so that $\omega = \ln W$ is distributed by F^{*k} for each value of k. Averaging this result over all possible values of k yields the distribution of logarithmic attenuation over a distance r:

$$G_r(\omega) = e^{-\alpha r} \sum_{k=0}^{\infty} \frac{(\alpha r)^k}{k!} F^{*k}. \tag{43}$$

As a result, attenuation precisely exhibits a log compound Poisson distribution (see (28)).

4.3.2 An analogy with the cascade of energy in turbulence

The parallel between such a cascade and the usual Richardson cascade using the interpretation of (28) of Section 3.2 is rather immediate. It suffices to compare the evolution of a ray of light travelling through a distribution of absorbing clouds to the evolution of turbulent dissipation through the scales.

Let us imagine that at each time and each position, we can see a turbulent flow as a collection of eddies of different sizes r. Note that r is no longer a position parameter but a scale parameter. Assume that the number of eddies at scales $r_2 < r < r_1$ follows a Poisson law with parameter $\Delta n_{r_1,r_2} = n(r_2) - n(r_1)$. The quantity $\Delta n(r_1, r_2)$ can be interpreted as an average number of 'dissipative structures' between scale r_1 and scale r_2. Note that the terms 'dissipative structures' have to be taken as an image without precise dynamical sense here. One may as well consider the quantity $-dn/d\ln r$ as a density of 'dissipative structures' by (logarithmic) unit scale (i.e., parameter α in Ambarzumian's model). Let us denote by W the energy transmission rate of each eddy and assume that W is a random variable with probability density function $\underline{F}$ that does not depend on the energy received by an eddy nor on its size r. Assume moreover that the W of eddies are independent and identically distributed. Then the rate of energy transfer from a larger scale r_1 to a smaller scale r_2 is described by a (log-)compound Poisson distribution (28). Furthermore, the distribution of the (log-)dissipation rate at some smaller scale r_2 can be deduced from the distribution of the (log-)dissipation rate at a larger scale r_1 thanks to a convolution with the propagator G of a compound Poisson cascade as in (39). The parameter Δn represents the average number of steps of the cascade: scale r_2 is reached after a random number k of steps from scale r_1. This random number follows a Poisson law and may vary with time and space. Similarly, the kernel G describes the elementary step which is determined by the distribution F. Exponents $\zeta(q)$ are a representation of the distribution F of the rate of transfer W of energy from an eddy to another.

We emphasize that this approach is not restricted to the abstract description of statistical features since IDC provide us with a way to the synthesis of modelling stochastic processes. Of course, we do not pretend that these processes are exactly reproducing all the properties of a real turbulent flow. However infinitely divisible cascades obey many interesting properties. They display both true stationarity as well as true continuous scaling. They are non Gaussian and scale invariant. They also have many degrees of freedom which can be used to introduce some geometry in the model. For instance, some controlled anisotropy at large or at small scales or a control of the small scale behaviour can be taken into account thanks to the integration kernel $f(\mathbf{x})$ — see Section 2.1. They can be defined in several dimensions. We emphasize again that the numerical synthesis is simple to implement. One could even think about working in 4 dimensions to describe the intermittent time evolution of some scalar field. Indeed, the "frozen hypothesis" which is often used in turbulence assumes that the measurement of the time evolution of some quantity at one point in a turbulent flow is equivalent to the measurement of this quantity along a streamline at fixed time (roughly $v(\mathbf{x},t) = v(\mathbf{x} - \boldsymbol{U}t, 0)$). Thus one expects some similarity between time fluctuations and spatial fluctuations. The synthesis of an IDC in 4 dimensions can be interpreted as the synthesis of an evolving multifractal scalar field (4D = 3D + t). Such stochastic processes may even be considered to simulate small scales in a purely stochastic manner in numerical simulations of turbulent flows. This is an open question.

4.4 Revisiting some classical models of turbulence

Relying on this vision of the cascade of energy in turbulence, we propose to visit again some traditional models of turbulence within the framework of infinitely divisible cascades.

4.4.1 Kolmogorov 1941

First of all, Kolmogorov's 1941 model corresponds to a perfect transfer of energy through the scales: dissipation occurs at an infinitely small scale only (as Re $\to \infty$). This corresponds to the case $F = \delta_0$ (i.e. constant multipliers $W = 1$) in the log Poisson model. Such a cascade is perfectly deterministic and exhibits no randomness: there is no cascade actually. This is somehow ironical since Richardson's cascade precisely inspired Kolmogorov's theory.

4.4.2 Log-Poisson (She-Lévêque) model

As a more flexible model still preserving scale invariance, then comes the log Poisson model. It corresponds to a situation where every dissipative structure transfer a constant fraction β of energy. Then each step of the cascade is deterministic since $W = \text{const.} = \beta$ and only the number of steps is random. The non-linearity scaling exponents appear as a consequence of a uniform distribution of dissipative effects at all scales.

It was shown in [20,46] that the log Poisson model corresponds to the She-Lévêque model [45] in which the special β value is associated to singular structures in the flow. A comparison between an experimental velocity signal from a turbulent flow and a synthetic signal built from an IDC of the Poisson type was presented in [16]. Both signal are quite similar. She and Waymire [46] even wondered whether it was possible to propose a classification of non linear dissipative systems on this basis. Each universality class would be defined by the number of singularities (each associated to a $\delta_{\ln\beta_i}$ in F) necessary to describe it by a log compound Poisson model with $F = \sum_i \delta_{\ln\beta_i}$. A classification would be obtained by simply 'counting the β_i'. At least two reasons explain that this idea did not persist. First, it assumes the existence of a finite number of singularities which is very restrictive in such a disordered system as a turbulent flow. Second, it is extremely difficult to support such a theoretical argument with experimental observations that do not provide sufficient accuracy to discriminate between different cases. Up to our knowledge, only very few assumptions can be imposed to the set of potentially relevant probability density functions F.

4.4.3 Compound Poisson cascades

The next natural generalization considers that the eddies are only *statistically* identical, i.e., their transmission ratios are no longer identical but are independently and identically distributed random variables. Such a framework is provided by log compound Poisson models (see Sect. 3) for which $F \neq \delta_{\ln\beta}$. In this case, the existence of a wide variety of dissipative structures with random dissipation rates is assumed. A new ingredient is provided by the possible fluctuation of the factor β within a continuum of values. The log Poisson model is recovered in this framework as the most simple case when F concentrates onto only one point ($F = \delta_{\ln\beta}$).

Beyond the work by She and Waymire [46], several authors have proposed log compound Poisson models. For instance, Castaing [9] used thermodynamical arguments to introduce a *temperature of turbulence* T that would reflect the efficiency of dissipative eddies. In fact, Castaing proposed a particular log compound Poisson model with $F(\omega) = e^{\omega/T}/T \Leftrightarrow \underline{F}(W) = 1/T\ W^{1/T-1}$. Then, both the number k of structures crossed between scales r_1 and r_2 and the rate of transfer distributed by F are random. In this work, Castaing proposed scaling exponents for velocity increments in turbulent flows of the form $\zeta(q) = q(1+3T)/3(1+qT)$. It is interesting to note that Yakhot [50] independently obtained exactly the same scaling exponents by studying the symmetries of Navier-Stokes equations (Yakhot introduced a parameter $B \equiv 1/T$): here is an evidence for potential links between multiplicative cascades and Navier-Stokes equation.

A marginal but interesting remark is that an infinitely divisible cascade has a priori no inverse, in agreement with the existence of a one-way direct (no inverse) cascade only as expected in turbulence. However, we remark that there is no objection to multipliers W taking values outside the interval [0, 1] thus permitting the existence of a local inverse cascade: $W > 1$ means that a small scale provides energy to a larger one. Infinitely divisible cascades may account for local inverse transfer of energy even though the cascade remains one-way on the average.

4.4.4 Kolmogorov 1962

Kolmogorov 1962 theory would correspond to a log-Normal cascade (G would be a Gaussian) that is a particular case of an IDC which is not a CPC since the Normal distribution is not compound Poisson [21]. As a consequence, while Kolmogorov 1962 theory inspired most of the work on multiplicative cascades, it is very difficult to synthesize log-Normal cascades numerically in d dimensions (the algorithm given in 3.4 is not suitable). This is due to the fact that a log-Normal cascade cannot be decomposed into the combination of a point process $(\mathbf{x}_i, r_i)$ on one hand and random multipliers on the other hand. Only the most general form of (7) using an independent random additive measure must be used. Thus, it is somewhat disappointing to remark that Kolmogorov 1962 theory is nice for theoretical and analytical computations but appears a very difficult to synthesize numerically in d dimensions, except for $d = 1$.

5 Conclusion

In this article, we have presented the generalization of the 1D infinitely divisible cascades [4,7,34,17,18,44] to d dimensions and described their main properties. The main statistical properties can be generalized in a quite natural manner. IDC provide us with a large class of non Gaussian scale invariant processes with controlled multifractal properties. The numerical synthesis of compound Poisson cascades (CPCs) is easy to implement. This is only one of the numerous nice features of CPCs. CPCs can as well help to better understand the notion of 'cascade of energy' in turbulence. To this aim, we referred to Ambarzumian's model to propose an analogy between the evolution of the energy of a ray of light through a distribution of absorbing clouds and the transfer of energy from large to small scales through a cascade of dissipative structures in a turbulent flow. This led us to propose a phenomenology that takes into account the main properties of turbulent flows altogether: scale invariance, statistical intermittency (i.e., departures from Gaussian distributions), the existence of coherent structures and an underlying multiplicative cascade. We emphasize that, in contrast with the way one usually imagine Richardson cascade, there is no discrete aspect left when one considers IDC in their most general form (7) and (8). This is somehow more difficult to imagine but maybe in better agreement with physical intuition.

Note that IDC in d dimensions may also be useful to synthesize a wide variety of multifractal scalar fields: textures in 2D, dynamical textures (3D = 2D + t), turbulent dissipation field in 3D, dynamical 3D scalar fields (4D = 3D + t, e.g., moving 3D cloud)... One may also explore the use of the integration kernel $f(\mathbf{x})$ to add some specific geometrical features in the model. Like in the 1D case [17], a non scale invariant version can be built. Software and demos are available from our webpage at `http://www.isima.fr/ chainais/SOFTWARE/`.

Several applications are yet the subject of ongoing work. IDC appear as a good model for the statistics of natural images [14]. A detailed study will be presented in a forthcoming paper [12]. The use of IDC to model images of the solar corona is under study in collaboration with the Royal Observatory of Belgium within the project CoSMIC (Corona of the Sun: modelling Images with Cascades). Despite much progress, the present work is still limited to scalar fields only. An important perspective is thus the quest for a vectorial model that would be able to reproduce the essential properties of a turbulent velocity field.

Parts of the results presented in this paper were communicated to GRETSI'05 on September 7, 2005 [15] and to ICIP'05 on September 14, 2005 [14].

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Appendix A: Power law scaling and control measure

This section aims at showing that the power law scaling behaviour of $Q_\ell(\mathbf{x})$ is linked to the choice of the control measure $dm(\mathbf{x}, r) = \dfrac{dr}{V_{1/2} r^{d+1}} d\mathbf{x}$ where $V_{1/2}$ is the volume of the sphere of radius 1/2 in d dimensions (e.g., $\pi/4$ in 2D). Note that we only focus on the basic definition of IDCs using the indicating function of a disk $\mathbb{1}_{\mathcal{D}}(\mathbf{x})$ as the geometrical kernel $f(\mathbf{x})$. Indeed, only the normalizing factor $V_{1/2}$ will change if another choice is made.

Since the moments of Q_ℓ are given by (12), the purpose here is to find (for fixed dimension d) the function $g(r)$ such that $dm(\mathbf{x}, r) = g(r) d\mathbf{x} dr$ and $m(\mathcal{C}_\ell) = -\log \ell$. We are looking for $g(r)$ such that:

$$\begin{aligned} m(\mathcal{C}_\ell(0)) &= \int_\ell^1 dr\ g(r) \int_{\mathcal{D}(0,r/2)} d^d\mathbf{x} \\ &= \int_\ell^1 dr\ g(r)\ V_{1/2}\ r^d \\ &= -\log \ell. \end{aligned} \quad (44)$$

This condition imposes the choice

$$g(r) = \frac{1}{V_{1/2}\; r^{d+1}}. \quad (45)$$

Appendix B: Autocovariance of log $Q_\ell(x)$

This section is devoted to the computation of the covariance $\mathrm{cov}(\log Q_\ell(0), \log Q_\ell(\mathbf{x}))$. Thus we need to compute $\mathbb{E}[\log Q_\ell(0)\log Q_\ell(\mathbf{x})]$ or equivalently $\mathbb{E}[M(\mathcal{C}_\ell(0))M(\mathcal{C}_\ell(\mathbf{x}))]$.

To this aim we use the following decomposition (see Fig. 1b):

$$\begin{aligned}\mathcal{J}(0,\mathbf{x}) &= \mathcal{C}_\ell(0)\cap\mathcal{C}_\ell(\mathbf{x}),\\ \mathcal{C}_\ell(0) &= \mathcal{L}(0,\mathbf{x})\cup\mathcal{J}(0,\mathbf{x}),\\ \mathcal{C}_\ell(\mathbf{x}) &= \mathcal{J}(0,\mathbf{x})\cup\mathcal{R}(0,\mathbf{x}).\end{aligned} \quad (46)$$

Moreover let

$$\begin{aligned}L(0,\mathbf{x}) &= M(\mathcal{L}(0,\mathbf{x})),\\ J(0,\mathbf{x}) &= M(\mathcal{J}(0,\mathbf{x})),\\ R(0,\mathbf{x}) &= M(\mathcal{R}(0,\mathbf{x})).\end{aligned} \quad (47)$$

Remark that $L(0,\mathbf{x})$, $J(0,\mathbf{x})$ and $R(0,\mathbf{x})$ are independent random variables since $\mathcal{L}(0,\mathbf{x})$, $\mathcal{J}(0,\mathbf{x})$ and $\mathcal{R}(0,\mathbf{x})$ are disjoint domains. Moreover, $L(0,\mathbf{x})$ and $R(0,\mathbf{x})$ are identically distributed since $m(\mathcal{L}(0,\mathbf{x})) = m(\mathcal{R}(0,\mathbf{x}))$. Thus we get:

$$\begin{aligned}\mathbb{E}[M(\mathcal{C}_\ell(0))M(\mathcal{C}_\ell(\mathbf{x}))] &= \mathbb{E}[L(0,\mathbf{x})]^2\\ &+ 2\mathbb{E}[L(0,x)]\mathbb{E}[J(0,\mathbf{x})] + \mathbb{E}[J(0,\mathbf{x})^2].\end{aligned} \quad (48)$$

For simple cones (circular, square...), one can show that for $|\mathbf{x}| \ll 1$:

$$m(\mathcal{J}(0,\mathbf{x})) = -\log|\mathbf{x}| + K + o(|\mathbf{x}|), \quad (49)$$

where K is some numerical constant depending on the chosen shape of the cone. Denoting by $C_1 = \varphi'(0)$, respectively $C_2 = \varphi''(0)$, which are the first and second cumulant of the distribution of $\log Q_\ell$, this yields:

$$\mathbb{E}[J(0,x)] = C_1\left[-\log|\mathbf{x}| + K + o(|\mathbf{x}|)\right]. \quad (50)$$

From

$$\begin{aligned}m(\mathcal{L}(0,\mathbf{x})) &= m(\mathcal{C}_\ell) - m(\mathcal{J}(0,x))\\ &= -\log\ell + \log|\mathbf{x}| - K + o(|\mathbf{x}|),\end{aligned} \quad (51)$$

we get:

$$\mathbb{E}[L(0,\mathbf{x})] = C_1\left[-\log\ell + \log|\mathbf{x}| - K + o(|\mathbf{x}|)\right]. \quad (52)$$

Combining (50) and (52) yields:

$$\begin{aligned}\mathbb{E}[L(0,\mathbf{x})]^2 + 2\mathbb{E}[L(0,\mathbf{x})]\mathbb{E}[J(0,\mathbf{x})] &= -C_1^2\left[(\log|\mathbf{x}|)^2\right.\\ &\left. -2K\log|x| + K^2 - (\log\ell)^2 + o(|\mathbf{x}|\log|\mathbf{x}|)\right].\end{aligned} \quad (53)$$

Furthermore, since M is an independently scattered additive infinitely divisible random measure we have:

$$\begin{aligned}\mathbb{E}[J(0,\mathbf{x})^2] &= \mathrm{var}(J(0,\mathbf{x})) + \mathbb{E}[J(0,\mathbf{x})]^2\\ &= -C_2\log|\mathbf{x}| + S + o(|\mathbf{x}|) + C_1^2\left[(\log|\mathbf{x}|)^2\right.\\ &\left.\quad -2K\log|\mathbf{x}| + K^2 + o(|\mathbf{x}|\log|\mathbf{x}|)\right].\end{aligned} \quad (54)$$

As a consequence, we obtain:

$$\mathbb{E}[M(\mathcal{C}_\ell(0))M(\mathcal{C}_\ell(\mathbf{x}))] = -C_2\log|\mathbf{x}| + S + o(|\mathbf{x}|) + C_1^2(\log\ell)^2. \quad (55)$$

Since $\mathbb{E}\log Q_\ell = \mathbb{E}M(\mathcal{C}_\ell) = -C_1\log\ell$, we finally get:

$$\mathrm{cov}\left(\log Q_\ell(0), \log Q_\ell(\mathbf{x})\right) = -C_2\log|\mathbf{x}| + S + o(|\mathbf{x}|) \quad (56)$$

where we recall that $C_2 = -\varphi''(0)$. See Figure 5b for an illustration.

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